

Stretch fast dynamo mechanism via conformal mapping in Riemannian manifolds

L.C. Garcia de Andrade¹

Abstract

Two new analytical solutions of self-induction equation, in Riemannian manifolds are presented. The first represents a twisted magnetic flux tube or flux rope in plasma astrophysics, which shows that the depending on rotation of the flow the poloidal field is amplified from toroidal field which represents a dynamo. The value of the amplification depends on the Frenet torsion of the magnetic axis of the tube. Actually this result illustrates the Zeldovich stretch, twist and fold (STF) method to generate dynamos from straight and untwisted ropes. Motivated by the fact that this problem was treated using a Riemannian geometry of twisted magnetic flux ropes recently developed (Phys Plasmas (2006)), we investigated a second dynamo solution which is conformally related to the Arnold kinematic fast dynamo. In this solution it is shown that the conformal effect on the fast dynamo metric only enhances the Zeldovich stretch, and therefore a new dynamo solution is obtained. When a conformal mapping is performed in Arnold fast dynamo line element a uniform stretch is obtained in the original line element. **PACS numbers:**

02.40.Hw-Riemannian geometries

¹Departamento de Física Teórica - Instituto de Física - UERJ
Rua São Fco. Xavier 524, Rio de Janeiro, RJ
Maracanã, CEP:20550-003 , Brasil.E-mail:garcia@dft.if.uerj.br.

I Introduction

Geometrical tools have been used with success [1] in Einstein general relativity (GR) have been also used in other important areas of physics, such as plasma structures in tokamaks as been clear in the Mikhailovskii [2] book to investigate the tearing and other sort of instabilities in confined plasmas [2], where the Riemann metric tensor plays a dynamical role interacting with the magnetic field through the magnetohydrodynamical equations (MHD). Recently Garcia de Andrade [3, 4] has also made use of Riemann metric to investigate magnetic flux tubes in superconducting plasmas. Thiffault and Boozer [5] following the same reasoning applied the methods of Riemann geometry in the context of chaotic flows and fast dynamos. Yet more recently Thiffeault [6] investigated the stretching and Riemannian curvature of material lines in chaotic flows as possible dynamos models. An interesting tutorial review of chaotic flows and kinematical dynamos has been presented earlier by Ott [7]. Also Boozer [8] has obtained a geomagnetic dynamo from conservation of magnetic helicity. Actually as pointed out by Baily and Childress [9] have called the attention to the fact that the focus nowadays the focus of kinematic dynamo theory is on the fast dynamos, specially in construction on curved Riemannian manifolds to stretch, twist and fold magnetic flows filaments or tubes to generated dynamo solutions. This method was invented by Zeldovich [10]. In this paper taking the advantage of the success of conformal Riemannian geometry techniques used to find out new solutions of Einstein general relativistic field equations[1], we use this same conformal geometrical technique to find new solutions of the incompressible flows in Arnold metric [11, 12, 13]. To resume , finding and being able to recognize the existence of dynamos or non-dynamos is not only important from physical and mathematical point of view, but also finding mathematical techniques which allows us to obtain new dynamo solution from dynamos or either nondynamo metrics such as the one we consider here as a flavour of the difficult to find a dynamo solution in Riemannian manifolds. This also motivates us to show that conformally Arnold fast dynamo metric, is also a dynamo solution. Recently Hanasz and Lesch [14] have used also a conformal Riemannian metric in \mathcal{E}^3 to investigate the galactic dynamo also using magnetic flux tubes. Also recently, Kambe [15] and Hattori and Zeitlin [16], have investigated the rate of stretching of Riemannian line elements of incompressible fluids, in the framework of differential geometry of diffeomorphisms, in the spirit of Zel-

dovich stretching. Actually they considered the exponential stretching of line elements in time, or dynamo action, in the context of negative curvature in turbulent flows. They also considered the concentration of vortex and magnetic flux tube. This provides a strong physical motivation to the present investigation. The paper is organized as follows: In section II the dynamo Riemann metric representing flux rope analytical solution of the self-induction equation in the case of zero resistivity is presented. In section III the dynamo solution in Riemannian conformal geometry is given. In section IV Riemannian curvature of a particular conformal dynamo is computed and in section V conclusions are presented.

II Thin flux rope dynamos in Riemannian manifold

In this section we shall consider generalization of the Riemann metric of a stationary twisted magnetic flux tube as considered by Ricca [17] in the Riemann manifold to address the nonstationary case where the toroidal and poloidal magnetic fields, in principle, may depend on time. With this metric at hand, we are to solve analytically the self-induction magnetic flow equation to check for the dynamo existence. Let us now start by considering the MHD field equations

$$\nabla \cdot \vec{B} = 0 \quad (1)$$

$$\frac{\partial}{\partial t} \vec{B} - \nabla \times [\vec{u} \times \vec{B}] - \eta \nabla^2 \vec{B} = 0 \quad (2)$$

where \vec{u} is a solenoidal field while η is the diffusion coefficient. Equation (2) represents the self-induction equation. The vectors \vec{t} and \vec{n} along with binormal vector \vec{b} together form the Frenet frame which obeys the Frenet-Serret equations

$$\vec{t}' = \kappa \vec{n} \quad (3)$$

$$\vec{n}' = -\kappa \vec{t} + \tau \vec{b} \quad (4)$$

$$\vec{b}' = -\tau \vec{n} \quad (5)$$

the dash represents the ordinary derivation with respect to coordinate s , and $\kappa(s, t)$ is the curvature of the curve where $\kappa = R^{-1}$. Here τ represents the

Frenet torsion. The gradient operator becomes

$$\nabla = \vec{t} \frac{\partial}{\partial s} + \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{e}_r \frac{\partial}{\partial r} \quad (6)$$

Now we shall consider the analytical solution of the self-induction magnetic equation investigated which represents a non-dynamo thin magnetic flux rope. Before the derivation of this result, we would like to point it out that it is not trivial, since the Zeldovich antidynamo theorem states that the two dimensional magnetic fields do not support dynamo action, and here as we shall see below, the flux tube axis possesses Frenet curvature as well as torsion and this last one cannot take place in planar curves. Let us now consider here the metric of magnetic flux tube

$$ds^2 = dr^2 + r^2 d\theta_R^2 + K^2(s) ds^2 \quad (7)$$

This is a Riemannian line element

$$ds^2 = g_{ij} dx^i dx^j \quad (8)$$

if the tube coordinates are (r, θ_R, s) [15] where $\theta(s) = \theta_R - \int \tau ds$ where τ is the Frenet torsion of the tube axis and $K(s)$ is given by

$$K^2(s) = [1 - r\kappa(s)\cos\theta(s)]^2 \quad (9)$$

Since we are considered thin magnetic flux tubes, this expression is $K \approx 1$ in future computations. Computing the Riemannian Laplacian operator ∇^2 in curvilinear coordinates [16] one obtains

$$\nabla^2 = \frac{1}{\sqrt{g}} \partial_i [\sqrt{g} g^{ij} \partial_j] \quad (10)$$

where $\partial_j := \frac{\partial}{\partial x^j}$ and $g := \det g_{ij}$ where g_{ij} is the covariant component of the Riemann metric of the flux rope. Here, to better compare the dynamo action generation of toroidal field from poloidal fields we shall consider the that the toroidal component of magnetic field $B_s(s)$ is given in the Frenet frame as

$$\vec{B}_s = b_0(s) \vec{t} \quad (11)$$

Note also that we have considered that the flux rope magnetic field does not depend on the r and θ_R coordinates. While the poloidal magnetic field the magnetic field here can be expressed as

$$\vec{B}_\theta(t, \theta) = e^{pt} b_1 \vec{e}_\theta \quad (12)$$

Now let us substitute the definition of the poloidal plus toroidal magnetic fields into the self-induction equation, which with the help of the expressions

$$\vec{e}_\theta = -\vec{n} \sin \theta + \vec{b} \cos \theta \quad (13)$$

$$\partial_\theta \vec{e}_\theta = -\vec{n}[(1 + \tau^{-1} \kappa) \sin \theta + \cos \theta] - \vec{b}[\cos \theta + \sin \theta] \quad (14)$$

and

$$\partial_t \vec{e}_\theta = \omega \vec{e}_\theta - \partial_t \vec{n} \sin \theta + \partial_t \vec{b} \cos \theta \quad (15)$$

Considering the equations for the time derivative of the Frenet frame given by the hydrodynamical absolute derivative

$$\dot{\vec{X}} = \partial_t \vec{X} + [\vec{v} \cdot \nabla] \vec{X} \quad (16)$$

where $\vec{X} = (\vec{t}, \vec{n}, \vec{b})$ represents the Frenet frame into the expressions for the total derivative of each Frenet frame vectors

$$\dot{\vec{t}} = \partial_t \vec{t} + [\kappa' \vec{b} - \kappa \tau \vec{n}] \quad (17)$$

$$\dot{\vec{n}} = \kappa \tau \vec{t} \quad (18)$$

$$\dot{\vec{b}} = -\kappa' \vec{t} \quad (19)$$

one obtains the values of respective partial derivatives of the Frenet frame as

$$\partial_t \vec{t} = -\tau \kappa [1 - \kappa \tau^{-2} \frac{v_\theta}{r}] \vec{n} \quad (20)$$

$$\partial_t \vec{n} = \tau \kappa [1 - \kappa \tau^{-2} \frac{v_\theta}{r}] \vec{t} + \frac{v_\theta}{r} \vec{b} \quad (21)$$

$$\partial_t \vec{b} = \kappa \tau^{-1} \frac{v_\theta}{r} \vec{n} \quad (22)$$

where we have used the hypothesis that $\dot{\vec{b}} = 0$ or $\kappa'(t, s) = 0$, which means that the curve curvature only depends on time. A simple example from solar

physics, would be a flux tube curved and with torsion oscillating with fixed sun spots. Substitution of these vectorial expressions into expression (15) yields

$$\partial_t \vec{e}_\theta = [\omega \cos \theta - \frac{v_\theta}{r}] \vec{b} - [\omega \sin \theta - \tau^{-1} \kappa \frac{v_\theta}{r}] \vec{n} + [\kappa \tau (1 + \tau^{-2} \frac{v_\theta}{r}) \sin \theta] \vec{t} \quad (23)$$

along with the equation

$$\frac{\partial B_\theta}{\partial s} = B_\theta r \tau \kappa \quad (24)$$

and the fact that $\frac{\partial B_s}{\partial s} = 0$, together with the self-induction equation we obtain the following system of equations, for a highly conductive fluid as our own universe, with resistivity $\eta = 0$

$$\partial_t B_\theta + \tau v_\theta \sin \theta B_\theta = 0 \quad (25)$$

$$\sin \theta B_\theta + \frac{B_s}{\sin \theta} [1 + \tau^{-2} \frac{v_\theta}{r}] = 0 \quad (26)$$

To obtain these last two expressions we assume that $v_\theta \gg v_s$ and that $\partial_s v_s = 0$ and that the continuity equation

$$\nabla \cdot \vec{v} = 0 \quad (27)$$

where we have considered that the flow is incompressible which is a reasonable approximation in plasma physics. This expression yields

$$\frac{\partial v_\theta}{\partial s} + v_\theta r \tau \kappa = 0 \quad (28)$$

Equation (25) can be rewritten as

$$[p + \tau v_\theta \sin \theta] = 0 \quad (29)$$

which upon substitution on the equation (26) yields

$$\frac{B_\theta}{B_s} \sin \theta [1 + \tau^{-2} \frac{v_\theta}{r}] = 0 \quad (30)$$

which with the assumption that the flux tube has a small twist and $v_\theta^2 \ll 1$. Now these equations

$$\frac{B_\theta}{B_s} = \frac{\tau \omega r}{p^2} \quad (31)$$

since $B_s \approx \text{constant} b_0$ by hypothesis, we have that relation (31) tells us that the relation $\frac{\tau \omega r}{p^2} > 0$ implies that the poloidal field is amplified from the constant modulus toroidal field, which is the dynamo rope condition. This physical situation happens often in the sun. Thus relation between the angular flow speed and the torsion and radius distance shows that there is a lower bound for the thin rope dynamo which is given by $r > \frac{p^2}{\omega \tau}$. The divergence-free equations for the magnetic and flow fields, allows us to write down the solutions for B_θ

$$B_\theta = B^0 e^{pt - \int \frac{r}{R} \cos \theta d\theta} \quad (32)$$

which if we recall the definition of the deviation of flat Riemann metric K of the tube above, we may express the integral in terms of $K(s)$ as

$$B_\theta = B^0 e^{pt - \int (1 - K(s)) d\theta} \quad (33)$$

which shows that in the very thin flux rope dynamo in this solution the effect curvature of the Riemannian tube is minor. Here we have used the fact that the external curvature of the rope is given by $\kappa_0 = \frac{1}{\kappa_0}$. The solution for torsion and velocity flow are essentially analogous.

III Conformal dynamos on manifolds

Conformal mapping on a Riemannian line element can in general be represented by

$$ds^2 = e^{2\lambda_0(\vec{x})} (ds_0)^2 \quad (34)$$

Manifolds related in this manner, are said conformally related. Note also that this is intrinsically connected to the stretch part of STF mechanism to generate rope dynamos. Actually in the previous section we considered a Riemannian metric for twisted flux tube which was stretched solely on the ds element along the magnetic axis of the dynamo rope, through the factor $K(s)$, of course when the flux rope is thin, the this stretch effect almost vanish though twist and fold may still be kept. Therefore strictly speaking, this is not a conformal mapping but only a stretch, therefore strictly speaking not all stretches are represented by conformal mapping but every conformal metric represents stretching in the Riemannian manifold. One of disadvantages of conformal stretching is that the stretching is uniform as in the case of Arnold

fast dynamo metric. Conformal metric techniques have also been widely used as a powerful tool obtain new solutions of the Einstein's field equations of GR from known solutions. By analogy, here we are using this method to yield new solutions of MHD dynamo from the well-known fast dynamo Arnold solution. We shall demonstrate that distinct physical features from the Arnold solution maybe obtained. Before that we just very briefly review the Arnold solution. The Arnold metric line element can be defined as [11]

$$ds^2 = e^{-2\lambda z} dp^2 + e^{2\lambda z} dq^2 + dz^2 \quad (35)$$

which describes a dissipative dynamo model on a 3D Riemannian manifold. By dissipative here, we mean that contrary to the previous section, the resistivity η is small but finite. The flow build on a toric space in Cartesian coordinates (p, q, z) given by $T^2 \times [0, 1]$ of the two dimensional torus. The coordinates p and q are build as the eigenvector directions of the toric cat map in \mathcal{R}^3 which possesses eigenvalues as $\chi_1 = \frac{(3+\sqrt{5})}{2} > 1$ and $\chi_2 = \frac{(3-\sqrt{5})}{2} < 1$ respectively. Note for example that if perform a simple constant conformal mapping such that

$$ds^2 = e^{2(\lambda z + \lambda_0)} dp^2 + e^{-2(\lambda z - \lambda_0)} dq^2 + e^{2\lambda_0} dz^2 \quad (36)$$

which represents a simple global translation and is not changed at every point in the manifold. Let us now recall the Arnold et al [13] definition of a orthogonal basis in the Riemannian manifold \mathcal{M}^3

$$\vec{e}_p = e^{\lambda z} \frac{\partial}{\partial p} \quad (37)$$

$$\vec{e}_q = e^{-\lambda z} \frac{\partial}{\partial q} \quad (38)$$

$$\vec{e}_z = \frac{\partial}{\partial z} \quad (39)$$

Assume a magnetic vector field \vec{B} on M

$$\vec{B} = B_p \vec{e}_p + B_q \vec{e}_q + B_z \vec{e}_z \quad (40)$$

The vector analysis formulas in this frame are

$$\nabla f = [e^{\lambda z} \partial_p f, e^{-\lambda z} \partial_q f, \partial_z f] \quad (41)$$

where f is the map function $f : \mathcal{R}^3 \rightarrow \mathcal{R}$. The Laplacian is given by

$$\Delta f = \nabla^2 f = [e^{2\lambda z} \partial_p^2 f + e^{-2\lambda z} \partial_q^2 f + \partial_z^2 f] \quad (42)$$

while the divergence is given by

$$\nabla \cdot \vec{B} = \text{div} \vec{B} = \text{div}[B_p \vec{e}_p + B_q \vec{e}_q + B_z \vec{e}_z] = [e^{\lambda z} \partial_p B_p + e^{-\lambda z} \partial_q B_q + \partial_z B_z] \quad (43)$$

In particular one may write

$$\text{div} \vec{e}_p = \text{div} \vec{e}_q = \text{div} \vec{e}_z = 0 \quad (44)$$

in turn the curl is written as

$$\text{curl} \vec{B} = \text{curl}[B_p \vec{e}_p + B_q \vec{e}_q + B_z \vec{e}_z] \quad (45)$$

where

$$\text{curl}_p \vec{B} = e^{-\lambda z} (\partial_q B_z - \partial_z (e^{\lambda z} B_q)) \quad (46)$$

$$\text{curl}_q \vec{B} = -e^{\lambda z} (\partial_p B_z - \partial_z (e^{-\lambda z} B_p)) \quad (47)$$

$$\text{curl}_z \vec{B} = e^{\lambda z} \partial_p B_q - e^{-\lambda z} \partial_q B_p \quad (48)$$

and

$$\text{curl} \vec{e}_p = -\lambda \vec{e}_q \quad (49)$$

$$\text{curl} \vec{e}_q = -\lambda \vec{e}_p \quad (50)$$

$$\text{curl} \vec{e}_z = 0 \quad (51)$$

The Laplacian operators of the frame basis are

$$\Delta \vec{e}_p = -\text{curl} \text{curl} \vec{e}_p = -\lambda^2 \vec{e}_p \quad (52)$$

$$\Delta \vec{e}_q = -\text{curl} \text{curl} \vec{e}_q = -\lambda^2 \vec{e}_q \quad (53)$$

$$\Delta \vec{e}_z = 0 \quad (54)$$

from these expressions Arnold et al [13] were able to build the self-induced equation in this Riemannian manifold as

$$\partial_t B_p + v \partial_z B_p = -\lambda v B_p + \eta [\Delta - \lambda^2] B_p - 2\lambda e^{\lambda z} \partial_p B_z \quad (55)$$

$$\partial_t B_q + v \partial_z B_q = +\lambda v B_q + \eta [\Delta - \lambda^2] B_q - 2\lambda e^{-\lambda z} \partial_q B_z \quad (56)$$

$$\partial_t B_z + v \partial_z B_z = \eta [\Delta - 2\lambda \partial_z] B_z \quad (57)$$

Decomposing the magnetic field on a Fourier series, Arnold et al were able to yield the following solution

$$b(p, q, z, t) = e^{\lambda v t} b(p, q, z - vt, 0) \quad (58)$$

where $B(x, y, z, t) = b(p, q, z, t)$ and the fast dynamo limit $\eta = 0$ was used. Now with these formulas, we are able to compute the solution of the self-induced magnetic equation in the background of conformal Riemannian line element

$$ds^2 = \Omega(z)[e^{-2\lambda z} dp^2 + e^{2\lambda z} dq^2 + dz^2] \quad (59)$$

The reason for using the general conformal stretching factor $\Omega(z)$ instead of the previous exponential stretching, is to show that the dynamo obtained is not only due to the exponential stretching but conformal dynamos, allow for the existence of more general conformal stretching. A far obvious, though important observation here is the fact that from the equations bellow, we recover the Arnold et al [13] if we simply make the conformal factor $\Omega := 1$. Denoting the dual one form for Arnold basis as

$$\phi_p = e^{-\lambda z} dp \quad (60)$$

$$\phi_q = e^{\lambda z} dq \quad (61)$$

$$\phi_z = dz \quad (62)$$

one obtains the Arnold fast dynamo metric as

$$ds^2 = \phi_p^2 + \phi_q^2 + \phi_z^2 \quad (63)$$

Thus the conformal one form dual basis can be expressed as

$$\phi_p^C = \Omega^{\frac{1}{2}} \phi_p \quad (64)$$

$$\phi_q^C = \Omega^{\frac{1}{2}} \phi_q \quad (65)$$

$$\phi_z^C = \Omega^{\frac{1}{2}} \phi_z \quad (66)$$

On the other hand the vector field basis in conformal metric becomes

$$\vec{e}_p = \Omega^{-\frac{1}{2}} e^{\lambda z} \frac{\partial}{\partial p} \quad (67)$$

$$\vec{e}_q = \Omega^{-\frac{1}{2}} e^{-\lambda z} \frac{\partial}{\partial q} \quad (68)$$

$$\vec{e}_z = \Omega^{-\frac{1}{2}} \frac{\partial}{\partial z} \quad (69)$$

Let us now repeat some of the fundamental vector analysis relations above in the conformal geometry. The first is the Laplacian of \vec{B}

$$\Delta_C \vec{B} = \Omega^{-1} \Delta \vec{B} - \frac{1}{2} \Omega^{-2} [\partial_z \Omega] \partial_z \vec{B} \quad (70)$$

A fundamental change here in the conformal stretching in Riemannian geometrical dynamos, is that of the velocity flow. In Arnold fast dynamo example, the flow is a very simple one which is given by $(0, 0, v)$ where v is constant. Here the dynamo flow is effectively in the dynamo equation by a term of the form

$$(\vec{v} \cdot \nabla) B_p = \Omega^{-1} v \partial_z B_p \quad (71)$$

which shows that a nonconstant effective velocity flow such as $v_{eff} = \Omega^{-1} v$ would act in conformal dynamos with respect to the previous Arnold example. The other fundamental component of the $curl[\vec{v} \times \vec{B}]$ given by

$$(\vec{B} \cdot \nabla) \vec{v} = \Omega^{-2} v [\partial_z \Omega] \vec{e}_z \quad (72)$$

With these expressions from conformal geometry in hand, we are now able to express the Arnold et al dynamo equations are

$$\partial_t \vec{B} + (\vec{v} \cdot \nabla_C) \vec{B} = (\vec{B} \cdot \nabla_C) \vec{v} + \eta \Delta_C \vec{B} \quad (73)$$

In terms of components this conformal self-induced equation in the Riemannian manifold can be expressed as

$$\partial_t B_p + \Omega^{-1} v \partial_z B_p = -\lambda \Omega^{-1} v B_p + \eta [(\Delta - \lambda^2) B_p - 2\lambda e^{\lambda z} \partial_p B_z] - \frac{\eta}{2} \Omega^{-2} (\partial_z \Omega) (\partial_z B_p + \lambda B_p) \quad (74)$$

The equation for q component can be obtained from p one by simply performing the substitution $\lambda \rightarrow -\lambda$. The expression for component- z is

$$\partial_t B_z + \Omega^{-1} v \partial_z B_z = \eta [\Delta - 2\lambda \partial_z - \frac{1}{2} \Omega^{-2} \partial_z \Omega \partial_z] B_z \quad (75)$$

Decomposing again the magnetic field on a Fourier series now in conformal geometry, yields the following solution

$$b(p, q, z, t) = e^{\lambda \Omega^{-1} vt} b(p, q, z - vt, 0) = e^{\lambda v_{eff} t} b(p, q, z - vt, 0) \quad (76)$$

where is the conformal Riemannian fast dynamo solution. As given explicitly in this solution the basic effect of the conformal geometry in fast dynamos is on the speed of dynamo which is an important physical effect.

IV Riemann curvature of conformal dynamos

The important role of negative curvature of geodesic flows in dynamos have been investigated by Anosov [19]. These Anosov flows, though somewhat artificial, provide excellent examples for numerical computation experiments in fast dynamos [20]. Within this motivation we include here a simple example of conformal spatially stretching, where $\Omega := e^{1\lambda z}$. This conformal stretching applied to Arnold metric yields the conformal metric as

$$ds^2 = dp^2 + e^{4\lambda z} dq^2 + e^{\lambda z} dz^2 \quad (77)$$

or in terms of the frame basis form ω^i ($i = 1, 2, 3$) is

$$ds^2 = (\omega^p)^2 + (\omega^q)^2 + (\omega_z)^2 \quad (78)$$

The basis form are write as

$$\omega^p = dp \quad (79)$$

$$\omega^q = e^{\lambda z} dq \quad (80)$$

and

$$\omega^z = e^{\frac{\lambda}{2} z} dz \quad (81)$$

By applying the exterior differentiation in this basis form one obtains

$$d\omega^p = 0 \quad (82)$$

$$d\omega^z = 0 \quad (83)$$

and

$$d\omega^q = \lambda e^{-\frac{\lambda}{2} z} \omega^z \wedge \omega^q \quad (84)$$

Substitution of these expressions into the first Cartan structure equations one obtains

$$T^p = 0 = \omega^p_q \wedge \omega^q + \omega^p_z \wedge \omega^z \quad (85)$$

$$T^q = 0 = \lambda e^{-\frac{\lambda}{2}z} \omega^z \wedge \omega^q + \omega^q_p \wedge \omega^p + \omega^q_z \wedge \omega^z \quad (86)$$

and

$$T^z = 0 = \omega^z_p \wedge \omega^p + \omega^z_q \wedge \omega^q \quad (87)$$

where T^i are the Cartan torsion 2-form which vanishes identically on a Riemannian manifold. From these expressions one is able to compute the connection forms which yields

$$\omega^p_q = -\alpha \omega^p \quad (88)$$

$$\omega^q_z = \lambda e^{-\frac{\lambda}{2}z} \omega^q \quad (89)$$

and

$$\omega^z_p = \beta \omega^p \quad (90)$$

where α and β are constants. Substitution of these connection form into the second Cartan equation

$$R^i_j = R^i_{jkl} \omega^k \wedge \omega^l = d\omega^i_j + \omega^i_l \wedge \omega^l_j \quad (91)$$

where R^i_j is the Riemann curvature 2-form. After some algebra we obtain the following components of Riemann curvature for the conformal antidynamo

$$R^p_{qpq} = \lambda e^{-\frac{\lambda}{2}z} \quad (92)$$

$$R^q_{zqz} = \frac{1}{2} \lambda^2 e^{-\lambda z} \quad (93)$$

and finally

$$R^p_{zpq} = -\alpha \lambda e^{-\frac{\lambda}{2}z} \quad (94)$$

We note that only component to which we can say is positive is R^p_{zqz} which turns the flow stable in this q-z surface. This component also dissipates away when z increases without bounds, the same happens with the other curvature components [21].

V Conclusions

In conclusion, we have used a well-known technique to find solutions of Einstein's field equations of gravity namely the conformal related spacetime metrics to find a new anti-dynamo solution in MHD three-dimensional Riemannian nonplanar flows. Examination of the Riemann curvature [21] components enable one to analyse the stretch and compression of the dynamo flow. New conformal fast dynamo metric are obtained from the conformally self-induced equation. It is shown that in the effect of conformal mapping in Riemannian dynamo flow is to change the fast dynamo speed. Future perspectives includes the investigation of homological obstructions in the conformal geodesic flows from Anosov flows generalizing the investigation of Vishik [22] and Friedlander and Vishik [23].

Acknowledgements

I would like to dedicate this paper to Professor Vladimir I. Arnold on the occasion of his senventh birthday. I would like also to thank CNPq (Brazil) and Universidade do Estado do Rio de Janeiro for financial supports.

References

- [1] H. Stephani et al, Exact solutions of Einstein field equations (2003) Cambridge university press. G. Ricci, Tensor Analysis, Boston.
- [2] A. Mikhailovskii, Instabilities in a Confined Plasma, (1998) IOP.
- [3] L. C. Garcia de Andrade, Physics of Plasmas 13, 022309 (2006).
- [4] L.C. Garcia de Andrade, Twist transport in strongly torsioned astrophysical flux tubes, Astrophysics and Space Science (2007) in press.
- [5] J. Thiffeault and A.H. Boozer, Chaos 11, (2001) 16.
- [6] J. Thiffeault, Stretching and Curvature of Material Lines in Chaotic Flows, (2004) Los Alamos arXiv:nlin. CD/0204069.
- [7] E. Ott, Phys. Plasmas 5, (1998) 1636.
- [8] A.H. Boozer, Phys. of Fluids B 5 (7), (1993) 2271.
- [9] S. Baily, S. Childress, Phys Rev Lett 59 (1987) 1573. S. Childress, A. Gilbert, Stretch, Twist and Fold: The Fast Dynamo (1996), Springer, Berlin.
- [10] Ya B. Zeldovich, A.A. Ruzmaikin and D.D. Sokoloff, The Almighty Chance (1990) World sci. Press.
- [11] V. Arnold and B. Khesin, Topological Methods in Hydrodynamics, Applied Mathematics Sciences 125 (1991) Springer.
- [12] V. Arnold, Ya B. Zeldovich, A. Ruzmaikin and D.D. Sokoloff, JETP 81 (1981), n. 6, 2052.
- [13] V. Arnold, Ya B. Zeldovich, A. Ruzmaikin and D.D. Sokoloff, Doklady Akad. Nauka SSSR 266 (1982) n6, 1357.
- [14] M. Hanasz, H. Lesch, Astronomy and Ap (2007), submitted.
- [15] T. Kambe, Geometrical theory of dynamical systems and fluid flows, (2000) world scientific, Singapore.

- [16] T. Kambe, Y. Hattori and V. Zeitlin, On the stretching of line elements in Fluids: an approach from differential geometry in Solar and Planetary dynamos,(1993) Newton Mathematical Institute Publications, Cambridge.
- [17] R. Ricca, Solar Physics 172 (1997),241.
- [18] W.D. D’haeseleer, W. Hitchon, J. Callen and J.L. Shohet, Flux Coordinates and Magnetic field Structure (1991) Springer.
- [19] D.V. Anosov, Geodesic Flows on Compact Riemannian Manifolds of Negative Curvature (1967) (Steklov Mathematical Institute, USSR) vol.90.
- [20] V. I. Arnold, E. Korkina, Vestnik Moscow state Univ. (1983) N3, 43.
- [21] E. Cartan, Riemannian geometry in an orthonormal Frame, (2001) Princeton University Press.
- [22] M. Vishik, Geophys. Astrophys. Fluid Dyn.48 (1989) 151.
- [23] S. Friedlander, M. Vishik, Chaos 1(2) (1991) 198.